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Article

The Homotopy Analysis Method for Strongly Nonlinear Initial / Boundary Value Problems

Mourad S. Semary and Hany N. Hassan [*]

Department of Basic Science, Benha Faculty of Engineering, Benha University, Benha, 13512, Egypt.

* Author to whom correspondence should be addressed; E-Mail: h_nasr77@yahoo.com; Tel.: +20 1225839389.

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Abstract: It is the purpose of the present paper to introduce an approach based on the homotopy analysis method to solve the nonlinear initial or boundary value problems with strongly nonlinear terms like (sqrt root, exp, sinh, cos,…). This approach reduces time consuming in the homotopy analysis method. Advantage of proposed idea is solving the problems without any transformation or approximation. The Sine-Gordon equation and some examples are used as illustrative examples to show the simplicity and effectiveness of the proposed approach. Also we solve the first extension of Bratu problem to show the proposed approach is capable to predict and calculate all branches of the solutions simultaneously.

Keywords: Homotopy analysis method; Nonlinear initial/ boundary value problems; The Sine-Gordon equation; The first extension of Bratu problem.

Mathematics Subject Classification 2000: 35C10, 65L10, 34L30.

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1. Introduction

The homotopy analysis method (HAM) [1–3] was first proposed by Liao in 1992 to solve many nonlinear problems. Liao first used the concept of homotopy to obtain analytic approximations of nonlinear equations, $N[u(x)]$ by means constructing so-called *the zero-order deformation equation*

$$(1-q)L[\phi(x,q)-u_0(x)] = q\hbar H(x)(N[\phi(x,q)]), \tag{1}$$

where $q \in [0, 1]$ is an embedding parameter, N is a nonlinear operator, $u(x)$ is an unknown function, and x denotes independent variable, $u_0(x)$ denotes an initial guess of the exact solution $u(x)$ which satisfies the initial or boundary conditions, $\hbar \neq 0$ an auxiliary parameter, $H(x)$ an auxiliary function and L an auxiliary linear operator. Obviously, we have $\phi(x, 0) = \beta$ when $q = 0$ and $\phi(x, 1) = u(x)$ when $q = 1$, respectively. The Taylor series of $\phi(x, q)$ with respect to the embedding parameter q reads

$$\phi(\boldsymbol{x}; q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x)\, q^m, \tag{2}$$

where

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x, q)}{\partial q^m}\bigg|_{q=0}. \tag{3}$$

Differentiating *the zero-order deformation equation* (1) m times with respective to the embedding parameter q and then dividing it by $m!$ and finally setting $q = 0$, we have the so-called mth-*order deformation equation*.

$$L[u_m(x) - \chi_m u_{m-1}(x)] = \hbar\, H(x) R_m\big(\vec{u}_{m-1}(x)\big) \tag{4}$$

where χ_m is defined by

$$\chi_m = \begin{cases} 0\,, m \leq 1 \\ 1, \;\; m > 1 \end{cases} \tag{5}$$

and

$$R_m\big(\vec{u}_{m-1}(x)\big) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x, q)]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{6}$$

The Mth-order approximation of $u(x)$ is given by

$$u(x) \cong U_M(x, \hbar) = \sum_{k=0}^{M} u_k(x) \tag{7}$$

In recent years determining approximate analytical solutions using the homotopy analysis method(HAM) has generated a lot of interest due to its applicability and efficiency [4-8]. Some modifications for different types of nonlinear equations have been developed in the literature [9-19]. In this paper we proposed an approach to improve and reduce time consuming in HAM for initial or boundary value problems with strong nonlinear terms terms like (sqrt root, exp, sinh, cos,…). The Sine-Gordon equation [13,20-21] and some examples are used as illustrative examples to show the simplicity and effectiveness of the proposed approach. Also we solve the first extension of Bratu problem [22] to show the proposed approach is capable to predict and calculate all branches of the solutions simultaneously. The numerical computation have done by Mathematica program by PC, CPU G620@2.60 GHz and 4GB of RAM.

2. The Proposed Approach

Consider the nonlinear initial or boundary value problems:

$$N[u(x)] = 0 \tag{8}$$

with initial or boundary conditions

$$\mathcal{B}\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad x \in \Gamma, \tag{9}$$

where N is a nonlinear operator, $\mathcal{B}$ is a boundary operator and Γ is a boundary of the domain Ω By choosing the initial guess $u_0(x) = \beta$, where β is any constant, we construct *the zero-order deformation equation* (1) as follows*:*

$$(1-q)L[\phi(x,q) - \beta] = q\hbar H(x)(N[\phi(x,q)]). \tag{10}$$

It is obvious that when the embedding parameter $q = 0$ and $q = 1$, Equation (10) becomes

$$\phi(x,0) = \beta \quad , \quad \phi(x,1) = u(x)\,. \tag{11}$$

Differentiating Equation (10) once time with respect to the embedding parameter q and setting $q = 0$, then equation (10) becomes

$$L[u_1(x)] = \hbar H(x)N[\beta] \quad , \tag{12}$$

taking the inverse linear operator(L^{-1}) of the both sides for the equation (12) becomes

$$u_1(x,\hbar) = L^{-1}\big[\hbar H(x)N[\beta]\big] \quad , \tag{13}$$

such that $\beta + u_1(x,\hbar)$ is satisfies the conditions (9). Differentiating equation (10) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$ and take the inverse linear operator(L^{-1}) of the both sides, then *the mth-order deformation equation* becomes

$$u_m(x,\hbar) = u_{m-1}(x) + L^{-1}[\hbar H(x)R(\vec{u}_{m-1})] \quad , \qquad m \geq 2 \tag{14}$$

subject to

$$\left.\frac{\partial^m \mathcal{B}\left(\phi(x,q), \frac{\partial \phi(x,q)}{\partial n}\right)}{\partial q^m}\right|_{q=0} = 0 \tag{15}$$

where $R(\vec{u}_{m-1})$ defined by equation (6). The high-order deformation equation. (14) obviously is just the ordinary differential equation with boundary condition (15) and, can be easily solved by using some symbolic software programs such as Mathematica or Maple. From equation (7), then The analytic approximation solution given by

$$u(x) \cong U_M(x,\hbar) = \sum_{m=0}^{M} u_m(x,\hbar) \ . \tag{16}$$

Equation (16) is a family of approximate solutions to the problem (8) in terms of the convergence-control parameter $\hbar$. By drawing $\hbar$-curve, we get the set $R_\hbar$. Using any $\hbar \epsilon R_\hbar$ one can get a convergent series solution.

3. Numerical Results

3.1. Example (1)

Consider the BVP with a hyperbolic sine nonlinearity [23]

$$u'''(x) - x\sinh(u) = 1 \ , \tag{17}$$

$$u(0) = 0, u(0.25) = 1, u(1) = 0 \ . \tag{18}$$

Firstly, we apply the standard homotopy analysis method on the problem (17). By choosing auxiliary linear operator L and initial guess $u_0(x)$ satisfies the boundary condition (18) as follows:

$$L[\phi(x,q)] = \frac{\partial^3 \phi(x,q)}{\partial x^3} \ , \tag{19}$$

And

$$u_0(x) = \frac{16}{3} x(1 - x). \tag{20}$$

We define a nonlinear operator as

$$N[\phi(x,q)] = \frac{\partial^3 \phi(x,q)}{\partial x^3} - x\sinh\big(\phi(x,q)\big) - 1 \ , \tag{21}$$

we can take $H(x) = 1$ and from equation (4) then *the mth-order deformation equation*

$$L[u_m(x) - \chi_m u_{m-1}(x)] = \hbar\, R_m(\vec{u}_{m-1}(x)), \tag{22}$$

with the boundary conditions for $m \geq 1$

$$u_m(0) = 0, u_m(.25) = 0, u_m(1) = 0 \tag{23}$$

where χ_m is defined by (5) and by equation (6). Thus $R_m(\vec{u}_{m-1}(x))$ is given by

$$R_m(\vec{u}_{m-1}(x)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1}\left(\frac{\partial^3 \phi(x,q)}{\partial x^3} - x \sinh \phi(x,q) - 1\right)}{\partial q^{m-1}} \right|_{q=0} . \tag{24}$$

Can be calculated $R_m(\vec{u}_{m-1})$(24) by using the definition(3) and then

$$R_1 = u_0'''(x) - x \sinh u_0(x) - 1 , \tag{25}$$

$$R_2 = u_1'''(x) - x u_1(x) \cosh u_0(x) , \tag{26}$$

$$R_3 = u_2'''(x) - \frac{x}{2}\left(\sinh(u_0(x)) u_1(x)^2 + 2\cosh(u_0(x)) u_2(x)\right) \tag{27}$$

and so on. According to the auxiliary linear operator L (19), the initial guess $u_0(x)$ (20) and R_1 then the first-order deformation equation (m=1) (22) becomes

$$u'''_1(x) = \hbar\left(u_0'''(x) - x \sinh\left(\frac{16}{3}x(1-x)\right) - 1\right). \tag{28}$$

The pro0blem (28) and (23) is a linear differential equation but require a very long time using the Mathematica to find $u_1(x)$, because the angle of "Sinh" is polynomial of the second degree. One can see equation R_3 (27), this the equation contain two strong functions are $Sinh(u_0(x))$ and $cosh(u_0(x))$ that means difficulty in obtaining $u_3(x)$. The proposed approach to prevent suffering by setting the initial guess $u_0(x) = \beta$, where β any constant so as to equation (28) as follows

$$u'''_1(x) = \hbar(-x \sinh \beta - 1) \tag{29}$$

The problem (28) converted to the linear differential equation (29) is very simple and can be easily solved by using the Mathematica and this leads to a reduction of time consumed in homotopy analysis method. Now we apply the proposed approach for the problem (17) and (18). We Choose the initial guess $u_0(x) = 0$. Then from equations (13) and (14), the higher order deformation equation (22) becomes for $m = 1$

$$u_1(x) = \hbar \int_0^x \int_0^\tau \int_0^\zeta -1 \, dt d\zeta d\tau + c_0 + c_1 x + c_2 x^2 \ , \tag{30}$$

where the integration constants c_0 , c_1 and c_2 are determined by the boundary conditions

$$u_1(0) = 0, \qquad u_1(0.25) = 1 \, , \, u_1(1) = 0 \tag{31}$$

and for $m \geq 2$

$$u_m(x) = u_{m-1}(x) + \hbar \int_0^x \int_0^\tau \int_0^\zeta R_m\big(\vec{u}_{m-1}(t)\big) \, dt d\zeta d\tau + c_0 + c_1 x + c_2 x^2 \ , \tag{32}$$

where c_0 , c_1 and c_2 are determined by the boundary conditions

$$u_m(0) = 0, \qquad u_m(0.25) = 0. \ , \ u_m(1) = 0. \tag{33}$$

We now give the solution of the higher order deformation equation at $m = 1$ and $m = 2$

$$u_1(x) = -\frac{1}{24}(-128 + \hbar)x - \frac{1}{24}(128 - 5\hbar)x^2 - \frac{\hbar x^3}{6},$$

$$u_2(x) = \frac{(-289408 - 215475\hbar)\hbar x}{5160960} + \frac{(1378944 + 1076915\hbar)\hbar x^2}{5160960} + \frac{(-860160 - 860160\hbar)\hbar x^3}{5160960} + \frac{(-458752 + 3584\hbar)\hbar x^5}{5160960} + \frac{(229376 - 8960\hbar)\hbar x^6}{5160960} + \frac{\hbar^2 x^7}{1260},$$

and so on. The approximation solution $U_M(x, \hbar)$ to the problem (17) and (18) is given by

$$u(x) \cong U_M(x, \hbar) = \sum_{\mathrm{m}=0}^{\mathrm{M}} \mathrm{u_m}(x, \hbar). \tag{34}$$

It is easy to discover the valid region of $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis (constant $U_8(0.5, \hbar)$ value) from Figure 1 that are

$R_\hbar \in [-0.6, -1.3]$. The absolute error is given by

$$Absolute\ error = |U_8(x, \hbar) - u_{Numerical}| \ , \tag{35}$$

where $u_{Numerical}$ obtained by Mathematica package to solve differential equations using "NDSolve" command. Table 1 shows The absolute errors (35) at different points in the interval $(0,1)$ when $\hbar = -1$. The results indicate the accuracy of the proposed approach.

Table 2 shows the CPU time consumed in calculating $u_m(x)$ for the problem (17) by HAM and the proposed approach. The proposed approach is powerful than HAM in saving consumed time as shown in table 1. We can calculate only the first order deformation equation u_1 using HAM. We waited one full hour to get $u_2(x)$ and did not get it, but the proposed approach can calculate the higher order deformation equation in a short time, for example u_8 consumed only 7.925 seconds.

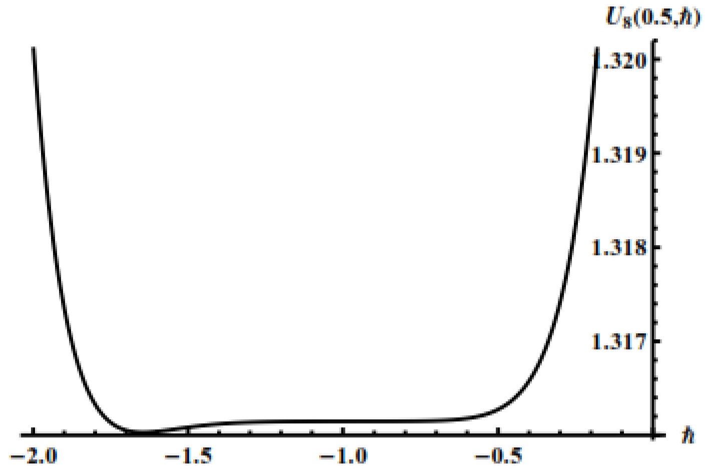

Figure 1. $\hbar$-curve for $U_8(0.5,\hbar)$ of the equation (34).

Table 1. The absolute errors (35) when $\hbar = -1$.

x	$U_8(x,\hbar)$	Absolute error (35)
0.1	0.48330167	1.217×10^{-7}
0.2	0.85535085	7.8492×10^{-8}
0.3	1.11726891	1.0678×10^{-7}
0.4	1.27035678	4.086×10^{-7}
0.5	1.31614726	8.1079×10^{-7}
0.6	1.25640126	1.2286×10^{-6}
0.7	1.09303594	1.488×10^{-6}
0.8	0.82799930	1.39459×10^{-6}
0.9	0.46312527	8.09235×10^{-7}

Table 2. The CPU time consumed in calculating $u_m(x)$ for example (1) by HAM and the proposed approach.

	u_1	u_2	u_4	u_6	u_8
HAM	2.512	N/A	N/A	N/A	N/A
The Proposed Approach	0.312	0.562	2.075	4.244	7.925

3.2. Example (2)

Consider the boundary value problem with a radical nonlinearity[23]

$$u'''(x) + \sqrt{1 - u^2(x)} = 0 \ , \tag{36}$$

$$u(0) = 0, \qquad u'(0) = 1, \qquad u\left(\frac{\pi}{2}\right) = 1 \ . \tag{37}$$

We have applied the standard homotopy analysis method on the problem and choosing the auxiliary linear operator Lin (19) and the initial guess $u_0(x)$ satisfying the boundary condition (37) as follows:

$$u_0(x) = x + \frac{4\,x^2}{\pi^2} - \frac{2\,x^2}{\pi}. \tag{38}$$

We define a nonlinear operator as

$$N[\phi(x,q)] = \frac{\partial^3 \phi(x,q)}{\partial x^3} + \sqrt{1 - \left(\phi(x,q)\right)^2} \ , \tag{39}$$

we can take $H(x) = 1$ and from equation (4) then *the mth-order deformation equation*

$$L[u_m(x) - \chi_m u_{m-1}(x)] = \hbar\, R_m\left(\vec{u}_{m-1}(x)\right), \tag{40}$$

with the boundary conditions for $m \geq 1$

$$u_m(0) = 0, u'_m(0) = 0, u_m\left(\frac{\pi}{2}\right) = 0 \tag{41}$$

where χ_m is defined by (5) and by equation (6). Thus $R_m\left(\vec{u}_{m-1}(x)\right)$ is given by

$$R_m\left(\vec{u}_{m-1}(x)\right) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}\left(\frac{\partial^3 \phi(x,q)}{\partial x^3} + \sqrt{1 + \left(\phi(x,q)\right)^2}\right)}{\partial q^{m-1}}\right|_{q=0} \tag{42}$$

Calculating $R_m(\vec{u}_{m-1})$ (42) using the definition (3), then

$$R_1 = u_0{}'''(x) + \sqrt{1 - \left(u_0(x)\right)^2}\,, \tag{43}$$

$$R_2 = u_1{}'''(x) - \frac{u_0(x) u_1(x)}{\sqrt{1 - (u_0(x))^2}} \ \ , \tag{44}$$

$$R_3 = u_2{}'''(x) - \frac{(u_0(x))^2 (u_1(x))^2}{2\left(1 - \left(u_0(x)\right)^2\right)^{3/2}} - \frac{(u_1(x))^2}{2\sqrt{1 - (u_0(x))^2}} - \frac{u_0(x) u_2(x)}{\sqrt{1 - (u_0(x))^2}} \ \ , \tag{45}$$

and so on. According to the auxiliary linear operator L (19), the initial guess $u_0(x)$ (38) and R_1 then the first-order deformation equation (m=1) (40) become

$$u'''_1(x) + \hbar \sqrt{1 - \left(x + \frac{4\,x^2}{\pi^2} - \frac{2\,x^2}{\pi}\right)^2} = 0. \tag{46}$$

The problem (46) and (41) is a linear differential equation but require a very long time using the mathematica to find $u_1(x)$. We waited one full hour to get $u_1(x)$ and did not get it. Now we apply the proposed approach to the problem (36) and (37) to see that the proposed approach reduced the time to find the higher order deformation equation (40). Choosing the initial guess $u_0(x) = 0$, from equations (13) and (14), the higher order deformation equation (40) becomes for $m = 1$

$$u_1(x) = \hbar \int_0^x \int_0^\tau \int_0^\zeta u_0''' + \sqrt{1 - (u_0)^2}\, dt d\zeta d\tau + c_0 + c_1 x + c_2 x^2 \ , \tag{47}$$

where the integration constants c_0 , c_1 and c_2 are determined by the boundary conditions

$$u_1(0) = 0, \quad u'_1(0) = 1 \ , u_1(\frac{\pi}{2}) \ = \ 1 \tag{48}$$

and for $m \geq 2$

$$u_m(x) = u_{m-1}(x) + \hbar \int_0^x \int_0^\tau \int_0^\zeta R_m\big(\vec{u}_{m-1}(t)\big)\, dt d\zeta d\tau + c_0 + c_1 x + c_2 x^2 \ , \tag{49}$$

where c_0 , c_1 and c_2 are determined by the boundary conditions

$$u_m(0) = 0, \qquad , u'_m(0) = 0 \quad u_m(\frac{\pi}{2}) \ = \ 0. \tag{50}$$

We now give the solution of the higher order deformation equation at $m = 1$ and $m = 2$

$$u_1(x) = x + \frac{1}{12}(\frac{48}{\pi^2} - \frac{24}{\pi} - \hbar\pi)x^2 + \frac{\hbar x^3}{6},$$

$$u_2(x) = -\frac{1}{12}\hbar(1+\hbar)\pi x^2 + \frac{1}{6}\hbar(1+\hbar)x^3,$$

and so on. The approximate solution $U_M(x, \hbar)$ to the problem (36) and (37) is given by

$$u(x) \cong U_M(x, \hbar) = \sum_{\mathrm{m}=0}^{\mathrm{M}} \mathrm{u_m}(x, \hbar). \tag{51}$$

It is easy to discover the valid region of $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis (constant $U_{12}(0.5, \hbar)$ value). From Figure 2 this is $R_\hbar \in [-0.6, -1.3]$. The absolute error is given by

$$absolute\ error = |U_{12}(x, \hbar) - u(x)| \ , \tag{52}$$

where $u(x)$ is the exact solution given by

$$u(x) = \sin(x) \tag{53}$$

Figure 3 shows the absolute errors (52) when $\hbar = -1$. The curve indicates the accuracy of the proposed approach. Table 3 shows the CPU time consumed in calculating $u_m(x)$ for the problem (36) by HAM and the proposed approach. We waited one full hour to get $u_1(x)$ using HAM and did not get it, but using the proposed approach we can calculate the higher order deformation equation in a short time, for example u_{12} consumed only 203.455 seconds.

Table 3. The CPU time consumed in calculating $u_m(x)$ for example (2) by HAM and the proposed approach

	u_1	u_2	u_4	u_6	u_8	u_{10}	u_{12}
HAM	N/A	N/A	N/A	N/A	N/A	N/A	N/A
The Proposed Approach	0.577	0.717	3.042	10.499	42.339	108.983	203.455

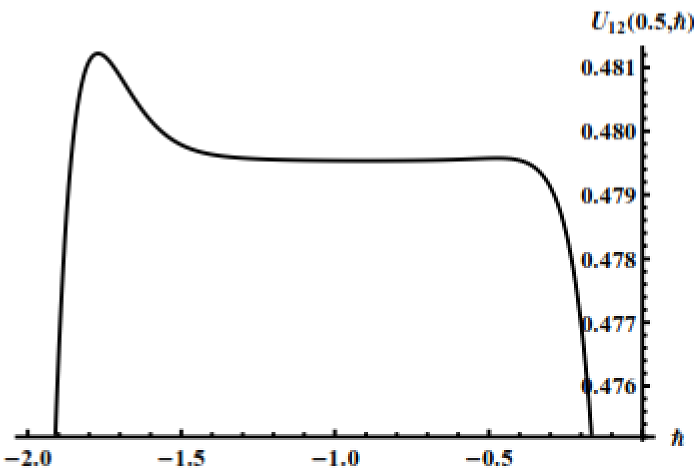

Figure 2. $\hbar$-curve for $U_{12}(0.5, \hbar)$ of the equation (51).

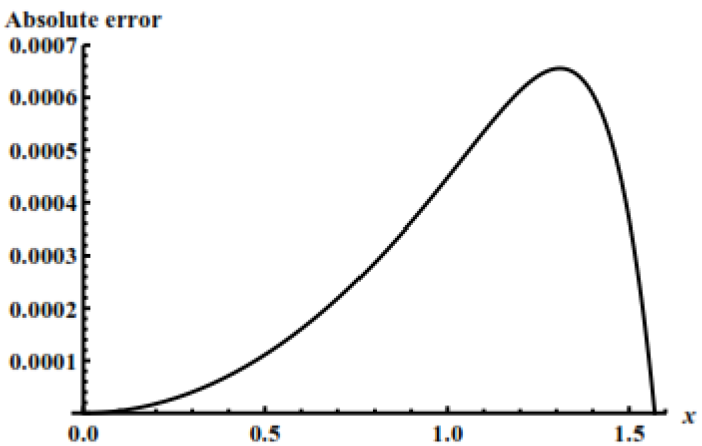

Figure 3. The absolute error (52) for example (2).

3.3. The Sine-Gordon Equation

We now apply the proposed approach for solving the nonlinear sine-Gordon equation [20,21] in the form:

$$u_{tt} - u_{xx} + \sin(u) = 0, \tag{54}$$

with initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 4\operatorname{sech}(x) \ . \tag{55}$$

The sine-Gordon equation (54) contains the strong nonlinear term $\sin(u)$. We apply the proposed approach on the problem by choosing an auxiliary linear operator and an initial guess $u_0(x,t)$ as follows

$$L[\phi(x,t,q)] = \frac{\partial^2 \phi(x,t,q)}{\partial t^2} \tag{56}$$

and

$$u_0(x,t) = 0 \quad . \tag{57}$$

Taking $H(x) = 1$, the first order deformation equation (13) becomes

$$u_1(x,t) = \hbar \int_0^t \int_0^\tau \int_0^\zeta R_1 \, dt d\zeta d\tau + c_0 + c_1 t + c_2 t^2 \ , \tag{58}$$

where the integration constants c_0 , c_1 and c_2 are determined by the boundary conditions

$$u_1(x,0) = 0, \qquad \left.\frac{\partial u_1(x,t)}{\partial t}\right|_{t=0} = 4\, sech(x) \tag{59}$$

and the higher order deformation equation (14) becomes for $m \geq 2$

$$u_m(x,t) = u_{m-1}(x,t) + \hbar \int_0^t \int_0^\tau \int_0^\zeta R_m\big(\vec{u}_{m-1}(x,t)\big) \, dt d\zeta d\tau + c_0 + c_1 t + c_2 t^2 \ , \tag{60}$$

where c_0 , c_1 and c_2 are determined by the boundary conditions

$$u_m(x,0) = 0, \qquad \left.\frac{\partial u_m(x,t)}{\partial t}\right|_{t=0} = 0 \tag{61}$$

$R_m(\vec{u}_{m-1}(x,t))$ can be calculated by using the definition (3), then

$$R_1 = \frac{\partial^2 u_0(x,t)}{\partial t^2} - \frac{\partial^2 u_0(x,t)}{\partial x^2} + \sin(u_0(x,t)) \tag{62}$$

$$R_2 = \frac{\partial^2 u_1(x,t)}{\partial t^2} - \frac{\partial^2 u_1(x,t)}{\partial x^2} - cos(u_0(x,t))u_1(x,t), \tag{63}$$

$$R_3 = \frac{\partial^2 u_2(x,t)}{\partial t^2} - \frac{\partial^2 u_2(x,t)}{\partial x^2} + \left(\frac{1}{2}\right)\left(-\sin\big(u_0(x,t)\big)u_1(x,t)^2 + \ 2\cos\big(u_0(x,t)\big)u_2(x,t)\right) \tag{64}$$

and so on. We now give the solution of the higher order deformation equation at $m = 1$ and $m = 2$

$u_1(x,t) = 4t \operatorname{sech}(x)$,

$u_2(x,t) = \frac{4}{3}\hbar t^3 (sech(x))^2$

and so on. The approximate solution $U_M(x,t,\hbar)$ to the problem (54) and (55) is given by

$$u(x,t) \cong U_M(x,t,\hbar) = \sum_{\mathrm{m}=0}^{\mathrm{M}} \mathrm{u_m}(x,t,\hbar). \tag{65}$$

The values of $\hbar$ in $R_\hbar \in [-0.6, -1.3]$ are found from the $\hbar$ -curve in figure 4. The absolute error is given by

$$absolute\ error = |U_M(x,t,\hbar) - u(x,t)| \ , \tag{66}$$

where $u(x,t)$ is the exact solution given by

$$u(x,t) = 4\, tan^{-1}\big(tsech(x)\big). \tag{67}$$

From tables 4 and 5 it is obvious that the proposed approach leads to a remarkable accuracy of the approximate solution. It is important to note that the accuracy of the solution obtained will be improved greatly if we increase the obtained terms. We can conclude that this method is more powerful for solving the sine Gordon equation. Finally, table 6 shows the CPU time consumed in calculating $u_m(x,t)$ by HAM and the proposed approach. We apply the standard homotopy analysis method, by choosing an auxiliary linear operator (56) and an initial guess $u_0(x,t) = \ 4t \operatorname{sech}(x)$. We waited one full hour to get $\boldsymbol{u_3(x)}$ using HAM and did not get it, but using the proposed approach we can calculate the higher order deformation equation in a short time, for example $\boldsymbol{u_7}$ consumed $\mathbf{343.154}$ seconds.

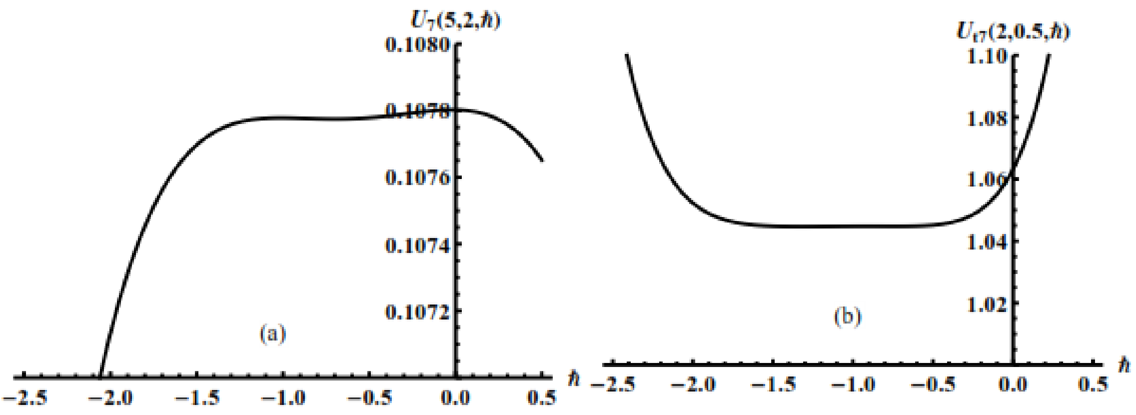

Figure 4. $\hbar$-curve for (a) $U_7(5,2,\hbar)$ (b) $U_{t7}(2,0.5,\hbar)$ of equation (65)

Table 4. The absolute error (66) of $U_3(x,t,\hbar)$ (65) at $\hbar=-1$.

x / t	0	0.1	1	5
0.02	1.7×10^{-9}	1.68×10^{-9}	4.64×10^{-10}	4.175×10^{-15}
0.05	1.66×10^{-7}	1.63×10^{-7}	4.53×10^{-8}	4.078×10^{-13}
0.08	1.73×10^{-6}	1.71×10^{-6}	4.75×10^{-7}	4.276×10^{-12}
0.1	5.27×10^{-6}	5.19×10^{-6}	1.44×10^{-6}	1.30×10^{-11}
0.3	1.17×10^{-3}	1.16×10^{-3}	3.46×10^{-4}	3.171×10^{-9}
0.5	1.29×10^{-2}	1.27×10^{-2}	4.33×10^{-3}	4.07×10^{-8}
0.8	9.42×10^{-2}	9.41×10^{-2}	4.27×10^{-2}	4.27×10^{-7}

Table 5. The absolute error (66) of $\boldsymbol{U_6(x,t,\hbar)}$ (65) at $\hbar=-1$.

x / t	0	0.1	1	5
0.02	1.80×10^{-16}	1.52×10^{-16}	6.93×10^{-18}	2.16×10^{-19}
0.05	6.21×10^{-13}	5.99×10^{-13}	2.87×10^{-14}	0
0.08	4.24×10^{-11}	4.10×10^{-11}	1.97×10^{-12}	1.73×10^{-18}
0.1	3.15×10^{-10}	3.04×10^{-10}	1.47×10^{-11}	1.21×10^{-17}
0.3	5.68×10^{-6}	5.49×10^{-6}	2.86×10^{-7}	8.42×10^{-14}
0.5	4.78×10^{-4}	4.64×10^{-4}	2.78×10^{-5}	6.40×10^{-12}
0.8	2.22×10^{-3}	7.67×10^{-2}	5.59×10^{-4}	1.03×10^{-10}

Table 6. The CPU time consumed in calculating $\boldsymbol{u_m(x,t)}$ for The Sine-Gordon equation (54) by HAM and the proposed approach.

	u_1	u_2	u_3	u_5	u_7
HAM	3.292	32.339	N/A	N/A	N/A
The Proposed Approach	0.311	0.451	1.496	10.077	343.154

3.4. The First Extension of Bratu Problem

We consider the first extension of Bratu problem [22] in form:

$$u''(x) + e^{u(x)} + e^{2u(x)} = 0 \quad , \tag{68}$$

with boundary conditions

$$u(0) = u(1) = 0 \quad . \tag{69}$$

Wazwaz studied the problem in 2012 using Adomian decomposition method and Padé approximants[22]. The result of the study is that the problem has dual solutions. In order to solve the problem (68) using the proposed approach, assume that $u(0.5) = \alpha$, then the boundary conditions (69) become

$$u(0) = 0,\ u(0.5) = \alpha \tag{70}$$

and

$$u(1) = 0 \quad . \tag{71}$$

We apply the proposed approach for the problem (68) and the boundary condition (70). By choosing an auxiliary linear operator L and an initial guess $u_0(x)$ as follows:

$$L[\phi(x,q)] = \frac{\partial^2 \phi(x,q)}{\partial x^2} \tag{72}$$

and

$$u_0(x) = 0 \quad . \tag{73}$$

Taking $H(x) = 1$, the first order deformation equation (13) becomes

$$u_1(x,\alpha) = \hbar \int_0^x \int_0^\tau R_1 \, dt d\tau + c_0 + c_1 x \quad , \tag{74}$$

where the integration constants c_0 and c_1 are determined by the boundary conditions

$$u_1(0) = 0, \qquad u_1(0.5) = \alpha \tag{75}$$

and the higher order deformation equation (14) becomes for $m \geq 2$

$$u_m(x,\alpha) = u_{m-1}(x,\alpha) + \hbar \int_0^x \int_0^\tau R_m\big(\vec{u}_{m-1}(t)\big) dt d\tau + c_0 + c_1 x \quad , \tag{76}$$

where c_0 and c_1 are determined by the boundary conditions

$$u_m(0) = 0, \qquad u_m(0.5) = 0 \tag{77}$$

$R_m(\vec{u}_{m-1}(x,t))$ can be calculated by using the definition (3), then

$$R_1 = u_0''(t) + e^{u_0(t)} + e^{2u_0(t)} \quad , \tag{78}$$

$$R_2 = u_1''(t) + e^{u_0(t)}u_1(t) + 2e^{2u_0(t)}u_1(t), \tag{79}$$

$$R_3 = u_2''(t) + (1/2)(e^{u_0(t)}u_1(t)^2 + 4e^{2u_0(t)}u_1(t)^2 + 2e^{u_0(t)}u_2(t) + 4e^{2u_0(t)}u_2(t)) \tag{80}$$

and so on. We now give the solution of the higher order deformation equation at $m = 1$ and $m = 2$

$$u_1(x,\alpha,\hbar) = \frac{1}{2}(-\hbar + 4\,\alpha)x + \hbar x^2$$

$$u_2(x,\alpha,\hbar) = \frac{1}{32}\hbar(-16 - 15\hbar - 8\,\alpha)x + \frac{1}{32}\hbar(32 + 32\hbar)x^2 + \frac{1}{32}\hbar(-8\hbar + 32\alpha)x^3 + \frac{\hbar^2 x^4}{4},$$

and so on. The approximate solution $U_M(x,\alpha,\hbar)$ to the problem (68) and (70) is given by

$$u(x) \cong U_M(x,\alpha,\hbar) = \sum_{\mathrm{m}=0}^{\mathrm{M}} \mathrm{u_m}(x,\alpha,\hbar). \tag{81}$$

Equation (81) is a family of approximate solutions to the problem (68) in terms of the convergence-control parameter $\hbar$ and α. Using the boundary condition(71), $u(1) = 0$, we find that:

$$u(1) \cong U_M(1,\alpha,\hbar) = 0. \tag{82}$$

We get α as a function of $\hbar$ from (82). This is plotted in Figure 5. From Figure 5, it is clear that two values of α, firstly lower solution at $\hbar$ interval $[-0.5, -1.5]$, secondly upper solution at $\hbar$ interval $[-1.2, -1.9]$. This example shows that the present method not only predict existence of multiple solution (two solutions) as shown in figure 5 by finding two constant values of α corresponding to two intervals of $\hbar$, but also calculate all branches of solution effectively without using one more initial approximation guess, one more auxiliary function and one more auxiliary linear operator. When $\hbar = -1$ we get the value of $\alpha = u(0.5) = 0.444153$ for lower branch solution and for upper branch solution when $\hbar = -1.8$, we get $\alpha = 1.07581$. the values of $u'(0)$ are1.5966 and 3.3846 for the lower and for upper branch solutions, respectively. The absolute error is given by

$$Absolute\ error = |U_{22}(x,\alpha,\hbar) - u_{Numerical}| \ , \tag{83}$$

where $u_{Numerical}$ obtained by Mathematica package to solve differential equations using “NDSolve” command and the absolute residual error is given by

$$\text{Absolute residual error} = \left|U_{22}''(x,\alpha,\hbar) + e^{U_{22}(x,\alpha,\hbar))} + e^{2U_{22}(x,\alpha,\hbar))}\right| \tag{84}$$

Table 7 shows the absolute error (83) for only lower solution against to numerical solution, because the Mathematica program detect only one solution to the problem (68) and this shows the importance of semi-analytic methods in this kind of problems. Table 6 shows the accuracy of the proposed approach in finding the lower solution of the problem (68). Figure 6 and figure 7 shows the absolute residual error (84) for the lower and upper solution. The Absolute residual error, indicating the accuracy of the approach used. Finally, figure 8 and figure 9 shows the lower and upper solution of the first extension of Bratu problem (68) obtained by the proposed approach.

Table 7. The absolute error (83) and $u(x)$ obtained by the proposed approach for the first extension of Bratu problem (68).

	Lower solution $\hbar = -1, \alpha = 0.444153$		Upper solution $\hbar = -1.8, \alpha = 1.07581$
x	Proposed approach $u(x)$	Absolute error (83)	Proposed approach $u(x)$
0.1	0.14884306037	5.04×10^{-10}	0.32658338150
0.2	0.27259259947	3.80×10^{-9}	0.61954042905
0.3	0.36599845458	1.62×10^{-9}	0.85886587377
0.4	0.42430886894	1.43×10^{-8}	1.01894634253
0.5	0.44415399569	3.58×10^{-8}	1.07581500664
0.6	0.42430886904	6.04×10^{-8}	1.01927870501
0.7	0.36599845447	7.43×10^{-8}	0.85949652183
0.8	0.27259259879	7.25×10^{-8}	0.62043963941
0.9	0.14884305891	6.88×10^{-8}	0.32777167385

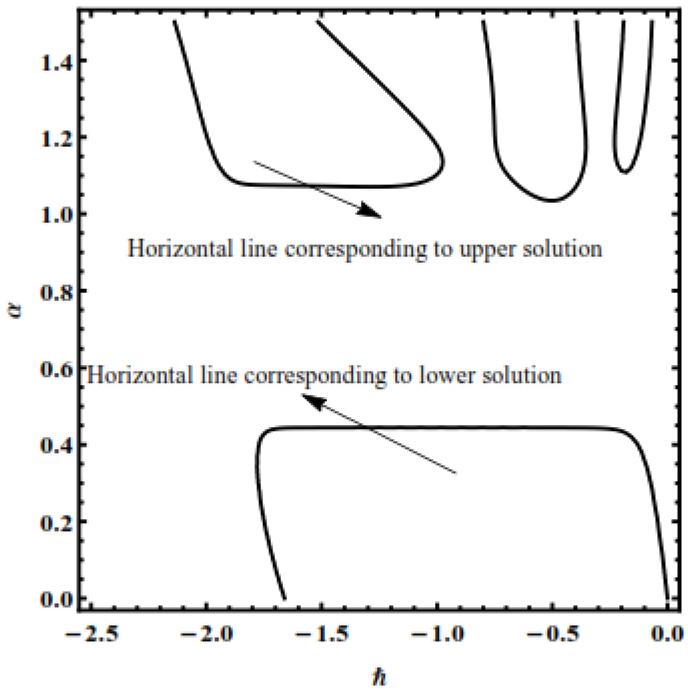

Figure 5. $\alpha - \hbar$ curve of equation (82) at $\boldsymbol{M = 22}$.

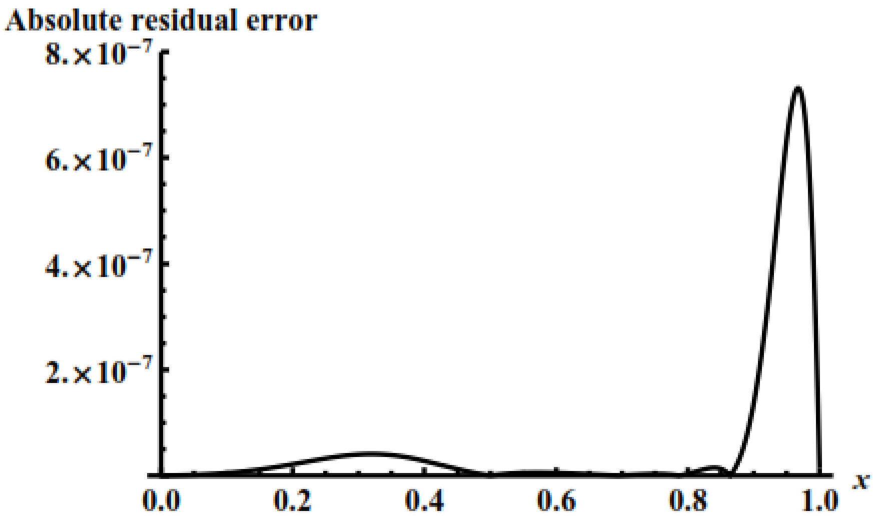

Figure 6. The absolute residual error (84) for lower solution of the first extension of Bratu problem (68).

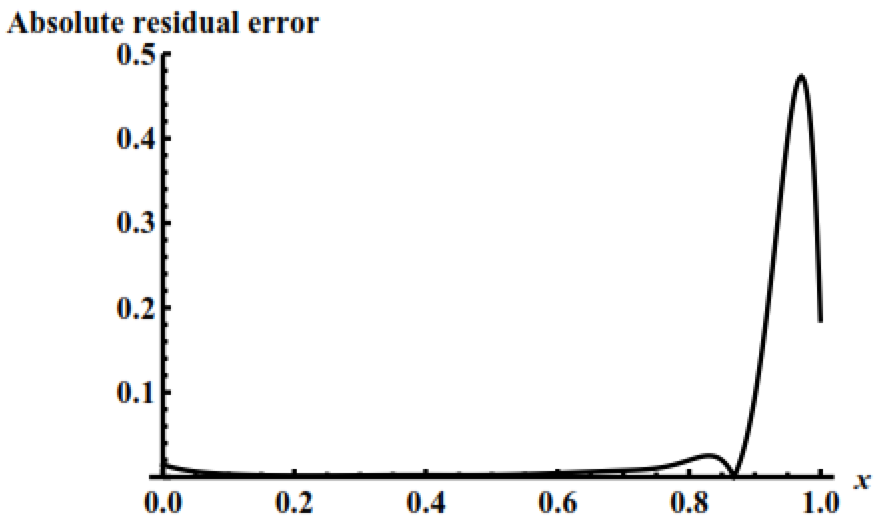

Figure 7. The absolute residual error (84) for lower solution of the first extension of Bratu problem (68).

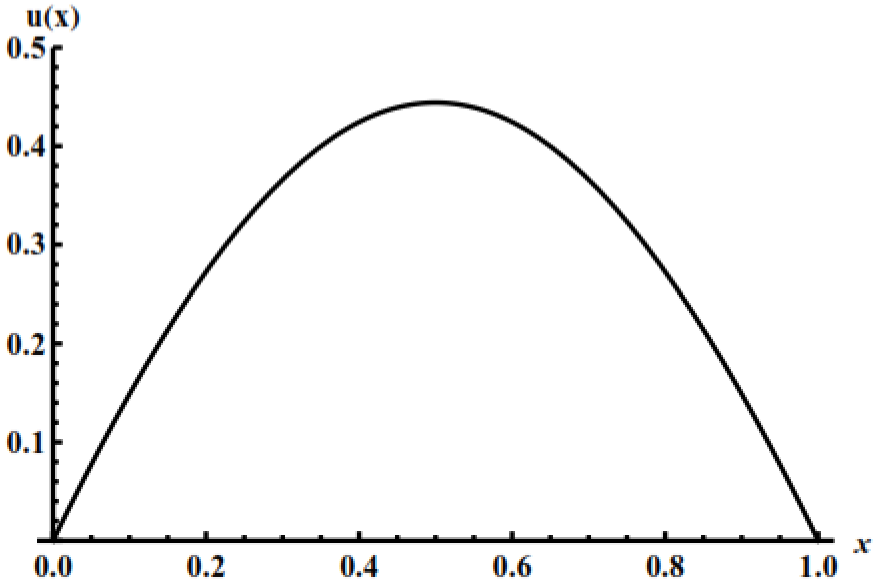

Figure 8. The lower solution of the first extension of Bratu problem (68) obtained by the proposed approach.

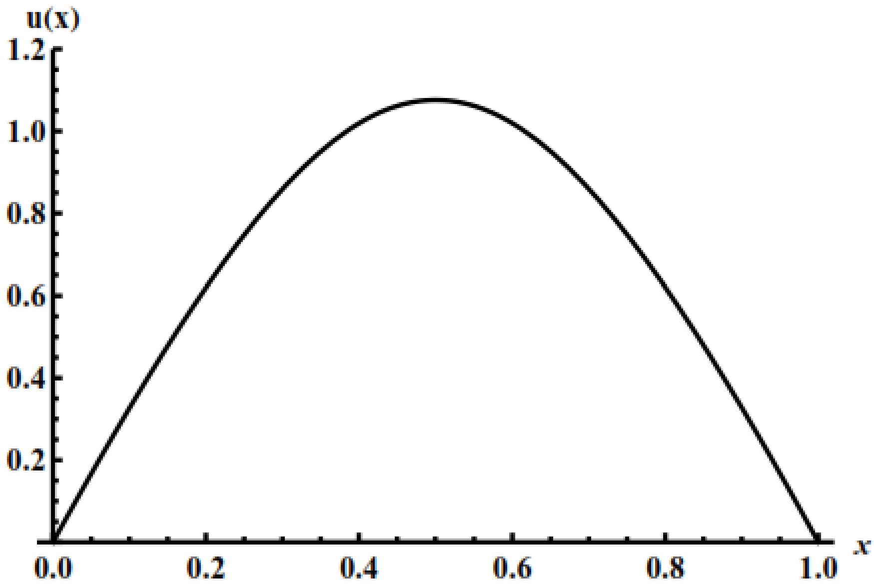

Figure 9. The upper solution of the first extension of Bratu problem (68) obtained by the proposed approach.

4. Conclusions

In this paper, we proposed an approach based on the homotopy analysis method to solve nonlinear initial or boundary value problems with strongly nonlinear terms. The proposed approach is to prevent suffering from the strongly nonlinear terms like (exp, sinh, cos,…) in the frame of the homotopy analysis method . We solve the problems without any transformation or approximation. The proposed approach succeeded in detecting dual solutions to the First extension of Bratu problem. It also reduces time consuming in the homotopy analysis method.

References

[1] Liao, S.-J. *The proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D thesis, Shanghai Jiao Tong University, Shanghai, **1992**.

[2] Liao, S.-J. *Beyond perturbation: introduction to the homotopy analysis method*. Chapman & Hall CRC/Press, Boca Raton **2003**.

[3] Liao, S.-J. Notes on the homotopy analysis method: some definitions and theorems. *Commun Nonlinear Sci Numer Simulat., 14* (**2009**): 983–97.

[4] Liao, S.-J. *Homotopy Analysis Method in Nonlinear Differential Equations*, Springer Heidelberg Dordrecht London New York, **2012**.

[5] Hassan, H. N., El-Tawil. M. A. An efficient analytic approach for solving two-point nonlinear boundary value problems by homotopy analysis method. *Math. Meth. Appl. Sci., 34* (**2011**): 977–989.

[6] Hassan, H. N., El-Tawil, M. A. Solving cubic and coupled nonlinear schrödinger equations using the homotopy analysis method. *Int. J. of Appl. Math and Mech., 7 (8)* (**2011**): 41-64.

[7] Hassan, H. N., El-Tawil, M. A. Series solution for continuous population models for single and interacting species by the homotopy analysis method. *Communications in Numerical Analysis*, (**2012**) doi:10.5899/2012/cna-00106.

[8] Hassan, H. N., Semary, M. S. Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method. *Communications in Numerical Analysis*, **2013.** doi.org/10.5899/2013/cna-00139.

[9] Bataineh, A. S., Noorani, M. S. M. and Hashim, I. On a new reliable modification of homotopy analysis method. *Commun. Nonlinear Sci. Numer. Simulat*, *14* (**2009**): 409-423.

[10] Bataineh, A. S., Noorani, M. S. M. and Hashim, I. Modified homotopy ananlysis method for solving system of second-order BVPs. *Commun. Nonlinear Sci. Numer. Simulat*., *14* (**2009**):430-442.

[11] Odibat, Z., Momani, S. and Hang, X. A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations. *Appl. Math. Modelling*, *34(* **2010**): 593-600.

[12] Alomari, A. K., Noorani, M. S. M. and Nazar, R. Homotopy Approach for the hyperchaotic Chen system. *Phys. Scr. 81* (**2010**) 045005 doi:10.1088/0031-8949/81/04/045005

[13] Hassan, H. N., El-Tawil, M. A. A new technique of using homotopy analysis method for solving high-order nonlinear differential equations. *Math. Meth. Appl. Sci*, *34* (**2011**): 728-742.

[14] Hassan, H. N., El-Tawil, M. A. new technique for using homotopy analysis method for second order nonlinear differential equations. *Appl. Math. Comput*, *219* (**2012**): 708–728.

[15] Liao, S.-J. An optimal homotopy analysis approach for strongly nonlinear differential equations. *Commun. Nonlinear Sci. Numer. Simulat.*, *15* (**2010**): 2003-2016.

[16] Niu, Z., Wang, C. one-step optimal homotopy analysis method for nonlinear differential equations. *Commun Nonlinear Sci Numer Simulat 15* (**2010**): 2026–2036.

[17] Abbasbandy,S., Shivanian., E. Predictor homotopy analysis method and its application to some nonlinear problems. *Commun. Nonlinear Sci. Numer. Simulat*, *16* (**2011**):2456-2468.

[18] Shukla, A. K., Ramamohan,T. R. and Srinivas, S. Homotopy Analysis Method with a Non-homogeneous term in the Auxiliary Linear Operator. *Commun. Nonlinear Sci. Numer. Simulat,17* (**2012**):3776-3787.

[19] Semary, M. S., Hassan, H. N. and El Naggar, K. The combined Homotopy analysis method and Laplace transform for solution MHD viscous flow due to a shrinking sheet. *Int. J. of Appl. Math and Mech.., 9 (18)* (**2013**):53-63**.**

[20] Yücel, U. Homotopy analysis method for the sine-Gordon equation with initial conditions. *Appl. Math. Comput*, *203* (**2008**): 387–395.

[21] Batiha, B. Noorani, M. S. M. and Hashim, I. Numerical solution of sine-Gordon equation by variational iteration method. *Physics Letters A*, *370* (**2007**): 437–440 .

[22] Wazwaz, A.-M. A reliable study for extensions of the Bratu problem with boundary conditions. *Math. Meth. Appl. Sci., 35* (**2012**): 845–856.

[23] Duan, J.-S., Rach. R. A. new modification of the A domain decomposition method for solving boundary value problems for higher order nonlinear differential equations. *Appl. Math. Comput*, *218* (**2011**): 4090-4118.